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Abstract
The rate $\nu_{1\to3}$ of one phonon decaying into three is calculated in the momentum range where one phonon decaying into two is forbidden in liquid ^{4}He. Exact and useful approximate expressions for the rate $\nu_{1\to3}$ are obtained. Its dependence on all parameters, and the physical reasons for these dependences, are analysed. It is shown that processes where one phonon scatters into three, and three phonons scatter to one phonon, rapidly establishes equilibrium in anisotropic and isotropic phonon systems. This shows that the momentum range, where one to three phonon processes are allowed, should be included in the subsystem of low-energy phonons where equilibrium occurs quickly, and not in the subsystem of high-energy phonons where spontaneous decay processes are forbidden and equilibrium occurs slowly.

1. Introduction

An important unresolved question concerning phonons in liquid ^{4}He, is 'what is the relaxation rate of phonons with energy in the range 8.94 K $\leqslant \varepsilon/k_B \leqslant$ 10 K'. This is the energy range above the energy where fast three-phonon processes, 3pp or $1 \to 2$, are allowed, and below the energy where the fastest rate is the relatively slow four-phonon process, 4pp or $2 \to 2$. In this range phonons can spontaneously decay into three or more phonons, and above this range no spontaneous decay is allowed. Experiments have clearly shown that a short pulse of phonons created in liquid ^{4}He separates, in a few tens of microseconds, into two groups, one comprises low-energy phonons, with typical energy 2 K, and the other high-energy phonons with energy $\geqslant$10 K. If the relaxation rate in the range 8.94 K $\leqslant \varepsilon/k_B \leqslant$ 10 K is high then these phonons must be included in the low-energy phonon group, but if the relaxation rate is

slow then they should be included in the high-energy phonon group. Where they are included has large consequences for the number of high-energy phonons with energy around 10 K, as well as for the thermodynamics of anisotropic phonon systems.

Phonon–phonon interactions in superfluid helium are crucially dependent on the relation between phonon energy ε and momentum p, which can be written as

$$\varepsilon = cp(1 + \psi(p)). \tag{1}$$

Here $c = 238$ m s^{-1} is sound velocity and $\psi(p)$ is a function which describes the deviation of a spectrum from linearity. Although $\psi(p)$ is small ($|\psi(p)| \ll 1$), it completely determines the mechanisms of phonon interactions.

When $p < p_c$ ($cp_c/k_B = 10$ K at zero pressure), the function $\psi > 0$, and the laws of conservation of energy and momentum allow spontaneous decay processes. The fastest of these processes is the three-phonon process, 3pp or $1 \rightarrow 2$, when one phonon decays into two or two interacting phonons combine into one. The 3pp is allowed up to $cp/k_B \leqslant 8.94$ K. When $p > p_c$ the function $\psi < 0$. In this case the dispersion is normal and processes of spontaneous phonon decay are forbidden by the laws of conservation of energy and momentum. Then the fastest processes are four-phonon processes, 4pp or $2 \rightarrow 2$, when there are two phonons in both the initial and the final states.

In the experiments [1] a unique phenomenon was observed: when one short current pulse was given to the heater, two pulses, well separated in time, were detected. The first was formed by low-energy phonons (l-phonons), and the second was formed by high-energy phonons (h-phonons). From the subsequent experiments (see [2, 3]), it was unambiguously demonstrated that the h-phonon pulse had not been created in the heater but was created by the l-phonon pulse during its motion from the heater to the detector in He II. The theory of this surprising phenomenon, when a rather cold l-phonon pulse with temperature close to $T_p \approx 1$ K creates high-energy phonons with energy $\varepsilon \geqslant 10$ K was given in [4, 5].

This theory was based on the fact that the rate of three-phonon processes $\nu_{1 \rightarrow 2}$ is some orders of magnitude higher than the rate of four-phonon processes $\nu_{2 \rightarrow 2}$. The strong inequality $\nu_{1 \rightarrow 2} \gg \nu_{2 \rightarrow 2}$ allowed us to consider that the phonons form two subsystems, one of low-energy phonons in which the relaxation occurs very quickly relative to all other times in the problem, i.e. $\tau_{1 \rightarrow 3} \ll t_p \ll t_{prop}$ where t_p and t_{prop} are the pulse length and propagation time respectively, and another of high-energy phonons in which the relaxation time $\tau_{2 \rightarrow 2}$ is $t_p < \tau_{2 \rightarrow 2} \ll t_{prop}$. In [4, 5] it was supposed that a subsystem of low-energy phonons is formed by phonons with $p < p_c$, and a subsystem of high-energy phonons is formed by phonons with $p > p_c$.

It was shown in [6–10], that three-phonon processes are not allowed by conservation laws all the way up to momentum p_c, where the function ψ vanishes, but only up to a momentum $\sqrt{4/5}\, p_c$ after which they are forbidden by the laws of conservation of energy and momentum. However, in the momentum range from $\sqrt{4/5}\, p_c$ to p_c processes of one phonon decaying into a greater number of phonons are still allowed. So at zero pressure, one phonon can decay into three up to a momentum $\tilde{p} = cp/k_B = 9.49$ K, decay into four is allowed up to 9.7 K, and into five up to 9.81 K and so on up to $\tilde{p}_c = 10$ K.

Therefore calculation of the rates of these processes will enable us to decide into which subsystem this region of momentum should be assigned. So if the rate of these processes is close to the rate of three-phonon processes this region of momenta should be assigned to l-phonons and if it appears less than $\nu_{2 \rightarrow 2}$ then it should be assigned to h-phonons and the boundary of the subsystems is at the point $\sqrt{4/5}\, p_c$. The answer to this question is very important as, according to [11], the rate of h-phonon creation with a given momentum, in an l-phonon pulse, is very sensitive to the numerical value of the momentum at the boundary between the l- and h-phonons.

In this paper one of the processes which does not conserve phonon number is investigated: it is the process of one phonon decaying into three. The rate of decay is calculated when there are no thermal phonons; this is the spontaneous decay rate. It is also calculated at finite temperature when there is additional stimulated decay. This is done for both isotropic distribution of thermal phonons and for strongly anisotropic phonon distributions. Exact and useful approximate equations for the decay rate are derived. The question of the numerical value of momentum, which delimits phonons of superfluid helium into l- and h-phonon subsystems, is considered and compared to experimental results.

2. The main characteristics of processes of one phonon decaying into three

The dispersion relation (1) is very important when studying the processes of one phonon decaying into three. For the calculations we need an analytical definition of the function $\psi(p)$. Here and below we use the simple analytical approximation which was obtained in [10] from [12] and is valid in the region of momentum $p \leqslant p_c$ which is of interest to us:

$$\psi(p) = 4\psi_{max} \frac{p^2}{p_c^2} \left(1 - \frac{p^2}{p_c^2} \right), \tag{2}$$

where $\psi_{max} = 0.046$ is the maximum value of the function $\psi(p)$ reached when $p = p_c/\sqrt{2}$. All threshold momentum values are calculated with approximation (2).

The maximum value of momentum p_{max} up to which processes of one phonon decaying into three are allowed can be obtained from the momentum and energy conservation laws

$$\mathbf{p}_1 = \mathbf{p}_2 + \mathbf{p}_3 + \mathbf{p}_4, \tag{3}$$

$$\varepsilon_1 = \varepsilon_2 + \varepsilon_3 + \varepsilon_4. \tag{4}$$

Restrictions on the momenta of interacting phonons are found from the relation between angles of phonons with momenta $\mathbf{p}_1$, $\mathbf{p}_2$ and $\mathbf{p}_3$, $\mathbf{p}_4$. From equations (1), (3) and (4) we have

$$\zeta_{12} = \frac{(2p_1 - 2p_2 + \phi)\phi - 2p_3 p_4 \zeta_{34}}{2p_1 p_2}, \tag{5}$$

where $\phi = f_1 - f_2 - f_3 - f_4$, and $f_i = p_i \psi(p_i)$. Also $\zeta_{ij} = 1 - \cos\theta_{ij}$ where θ_{ij} is the angle between phonons with momenta $\mathbf{p}_i$ and $\mathbf{p}_j$.

Putting the values of ζ_{34} and ζ_{12} equal to zero in (5), we get the equation defining the boundaries of the regions in which processes of one phonon decaying into three can take place. Substituting (2) in (5) with $\zeta_{34} = 0$ and $\zeta_{12} = 0$ we have, in the first approximation of small ψ, an equation in the fourth power of $p_3(p_1, p_2)$. Its solutions are

$$p_{3\pm}(p_1, p_2) = \frac{1}{10} \left(5p_1 - 5p_2 \pm \sqrt{5}\sqrt{-15p_1^2 + 10p_1 p_2 - 15p_2^2 + 12p_c^2} \right), \tag{6}$$

$$p_3(p_1, p_2) = p_1.$$

The fourth solution $p_3(p_1, p_2) = -p_2$ should be omitted as it does not satisfy the condition $p_3 \geqslant 0$.

The equation (6) defines the boundaries of the region in which processes of one phonon decaying into three can take place. From the positivity of the radicand in (6) the restriction on the momentum range of the second phonon can be obtained:

$$\max(0, p_{2-}) < p_2 < \min(p_1, p_{2+}), \tag{7}$$

where

$$p_{2\pm}(p_1) = \frac{1}{3} \left(p_1 \pm 2\sqrt{2}\sqrt{p_{max}^2 - p_1^2} \right), \qquad p_{max} = \sqrt{\frac{9}{10}} p_c. \tag{8}$$

It follows from equations (3)–(8) that the values of momenta of phonons participating in processes $1 \to 3$ can lie in the range (see also [13, 14])

$$0 < p_{2,3,4} < \sqrt{\tfrac{3}{5}} p_c = p_{\min}, \tag{9}$$

$$0 < p_1 < p_{\max}. \tag{10}$$

At the saturated vapour pressure we have $\tilde{p}_{\min} = 7.75$ K and $\tilde{p}_{\max} = 9.49$ K.

We note that the processes of one phonon decaying into three in the region of momenta

$$\sqrt{\tfrac{4}{5}} p_c < p_1 < \sqrt{\tfrac{9}{10}} p_c, \tag{11}$$

only involve small angles: from equations (3)–(5), we see that the angle between the momentum of the initial phonon and the momentum of any of the created phonons does not exceed 23°.

The interaction of phonons in superfluid helium is described by the Landau Hamiltonian which we write as (see, for example, [15–17])

$$\hat{H}_{ph} = \hat{H}_0 + \hat{V}_3 + \hat{V}_4. \tag{12}$$

Here $\hat{H}_0$ is the Hamiltonian of noninteracting phonons and terms $\hat{V}_3$ and $\hat{V}_4$ describe the interaction of phonons, caused by the third and fourth orders of small deviations of the system from equilibrium respectively.

We can use the hydrodynamic Hamiltonian (12) for phonons with energy $\varepsilon / k_B \approx 10$ K because the de Broglie wavelength of an atom in the liquid ^{4}He is larger than the atomic spacing. The application of the hydrodynamic approach to small wavelengths was discussed in [18]. In this approach the delocalization of the atoms in the quantum liquid allows us to use the relation (1) with constant c, for phonons of any energy.

The probability density for the process of one phonon decaying into three can be written as

$$W(\mathbf{p}_1 | \mathbf{p}_2 \mathbf{p}_3 \mathbf{p}_4) = \frac{2\pi}{\hbar} V^2 |H_{fi}|^2 \frac{1}{(2\pi \hbar)^6}. \tag{13}$$

Here V is the volume of the system and H_{fi} is the amplitude of the process in which one phonon in the initial state decays into three phonons in the final state. This is obtained in second order perturbation theory on $\hat{V}_3$ and in first order perturbation theory on $\hat{V}_4$, with the help of standard procedures (see, for example, [11, 15–17, 19, 20]) i.e.

$$H_{fi} = \sum_Q \frac{\langle \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4 | \hat{V}_3 | \mathbf{Q} \rangle \langle \mathbf{Q} | \hat{V}_3 | \mathbf{p}_1 \rangle}{E_i - E_{\mathbf{Q}}} + \langle \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4 | \hat{V}_4 | \mathbf{p}_1 \rangle, \tag{14}$$

where $\mathbf{Q}$ is an intermediate state with energy $E_{\mathbf{Q}}$ and E_i is the energy of the initial state.

When a phonon with momentum $\mathbf{p}_1$ decays into three having momenta $\mathbf{p}_2$, $\mathbf{p}_3$, $\mathbf{p}_4$, six intermediate states I–VI in which phonons have momenta

I, $\mathbf{p}_1 - \mathbf{p}_2, \mathbf{p}_2$; II, $\mathbf{p}_1 - \mathbf{p}_3, \mathbf{p}_3$; III, $\mathbf{p}_1 - \mathbf{p}_4, \mathbf{p}_4$;

IV, $\mathbf{p}_1, \mathbf{p}_3, \mathbf{p}_4, -\mathbf{p}_3 - \mathbf{p}_4$; V, $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_4, -\mathbf{p}_2 - \mathbf{p}_4$; VI, $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, -\mathbf{p}_2 - \mathbf{p}_3$ (15)

are possible. From equations (14) and (15) we have

$$H_{fi} = \delta_{\mathbf{p}_1; \mathbf{p}_2 + \mathbf{p}_3 + \mathbf{p}_4} \frac{\sqrt{p_1 p_2 p_3 p_4}}{8 \rho V} M. \tag{16}$$

Here $\rho = 145$ kg m^{-3} is a density of He II,

$$M = M_{12}^{(2)} + M_{13}^{(2)} + M_{14}^{(2)} + M_{12}^{(4)} + M_{13}^{(4)} + M_{14}^{(4)} + M_4 \tag{17}$$

is the matrix element which consists of seven terms, six of which correspond to six intermediate states (15), and the seventh is defined by first-order perturbation theory on $\hat{V}_4$. The superscripts

show the number of phonons in the intermediate state. From the explicit definitions of the terms in the Hamiltonian, [15], we derive the matrix elements as

$$M_{12}^{(2)} = \frac{\varepsilon_{1-2}}{\varepsilon_1 - \varepsilon_2 - \varepsilon_{1-2}} \left(2u - 1 + \mathbf{n}_3\mathbf{n}_4 + \mathbf{n}_3\mathbf{n}_{3+4} + \mathbf{n}_4\mathbf{n}_{3+4} \right)$$
$$\times \left(2u - 1 + \mathbf{n}_1\mathbf{n}_2 + \mathbf{n}_1\mathbf{n}_{1-2} + \mathbf{n}_2\mathbf{n}_{1-2} \right), \tag{18}$$

$$M_{12}^{(4)} = -\frac{\varepsilon_{1-2}}{\varepsilon_1 - \varepsilon_2 + \varepsilon_{1-2}} \left(2u - 1 + \mathbf{n}_3\mathbf{n}_4 - \mathbf{n}_3\mathbf{n}_{3+4} - \mathbf{n}_4\mathbf{n}_{3+4} \right)$$
$$\times \left(2u - 1 + \mathbf{n}_1\mathbf{n}_2 - \mathbf{n}_1\mathbf{n}_{1-2} - \mathbf{n}_2\mathbf{n}_{1-2} \right), \tag{19}$$

$$M_4 = 4\{(u - 1)^2 + w\}, \tag{20}$$

where $\mathbf{n}_i = \frac{\mathbf{p}_i}{p_i}$, $\varepsilon_i = \varepsilon(p_i)$, $\varepsilon_{i-j} = \varepsilon(|\mathbf{p}_i - \mathbf{p}_j|)$, $u = \frac{\rho}{c}\frac{\partial c}{\partial \rho} = 2.84$ is the Grüneisen constant and $w = \frac{\rho^2}{c}\frac{\partial^2 c}{\partial \rho^2} = 0.188$. Other terms of (17), could be obtained from (18) and (19) by the replacement of corresponding subscripts. We notice that the first three terms of expression (17) are resonant: when $\psi(p) = 0$ their denominators can vanish, leading to an essential divergence of the matrix elements. These three terms give the main contribution to expression (17). The remaining terms give approximately the same small contribution.

3. The kinetic equation for the processes of one phonon decaying into three phonons

The kinetic equation describing the change of the distribution function $n_1 \equiv n(\mathbf{p}_1)$ of a phonon with momentum $\mathbf{p}_1$ due to processes of one phonon decaying into three can be written as

$$\frac{dn_1}{dt} = \frac{1}{3!} \int W(\mathbf{p}_1|\mathbf{p}_2\mathbf{p}_3\mathbf{p}_4)\{n_2n_3n_4(1 + n_1) - n_1(1 + n_2)(1 + n_3)(1 + n_4)\}$$
$$\times \delta(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4)\delta(\mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4)\, d^3 p_2\, d^3 p_3\, d^3 p_4$$
$$+ \frac{1}{2} \int W(\mathbf{p}_4|\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3)\{n_4(1 + n_1)(1 + n_2)(1 + n_3) - n_1n_2n_3(1 + n_4)\}$$
$$\times \delta(\varepsilon_4 - \varepsilon_1 - \varepsilon_2 - \varepsilon_3)\delta(\mathbf{p}_4 - \mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3)\, d^3 p_2\, d^3 p_3\, d^3 p_4. \tag{21}$$

Here the first term corresponds to a phonon with a momentum $\mathbf{p}_1$ decaying into three phonons and the inverse process and the second term corresponds a phonon with momentum $\mathbf{p}_1$ combining with other phonons and process inverse to it.

In the region of momentum (11) which is of interest to us the second term in the kinetic equation vanishes due to the laws of conservation of energy and momentum. In this case the kinetic equation can be written as

$$\frac{dn_1}{dt} = N_b - N_d. \tag{22}$$

Here

$$N_{b,d} = \frac{1}{3!} \int W(\mathbf{p}_1|\mathbf{p}_2\mathbf{p}_3\mathbf{p}_4)n_{b,d}\delta\left(\varepsilon_{\sum}\right)\delta\left(\mathbf{p}_{\sum}\right) d^3 p_2\, d^3 p_3\, d^3 p_4, \tag{23}$$

$$n_b = n_2n_3n_4(1 + n_1), \qquad n_d = n_1(1 + n_2)(1 + n_3)(1 + n_4), \tag{24}$$

$$\varepsilon_{\sum} = \varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4, \qquad \mathbf{p}_{\sum} = \mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4. \tag{25}$$

As shown in [10, 21] when $p_1 < \sqrt{4/5}\, p_c$, fast three-phonon processes form a local-equilibrium distribution function which can be written as

$$n^{(0)}(\mathbf{p}_l) = \left\{ \exp\left(\frac{\varepsilon_l - \mathbf{p}_l\mathbf{u}}{k_B T}\right) - 1 \right\}^{-1}, \tag{26}$$

where

$$\mathbf{u} = \mathbf{N}c(1 - \chi) \tag{27}$$

is a drift velocity, $\mathbf{N}$ is a unit vector directed along the total momentum of phonon system, which defines the anisotropy axis of phonon system and χ is the anisotropy parameter.

In isotropic phonon systems $\chi = 1$. In the case corresponding to experiments [1–3], phonon pulses are strongly anisotropic phonon systems with $\chi \ll 1$.

In calculations it will be more convenient to use the expression

$$n^{(0)}(p_l, \zeta_l) = \left\{ \exp \left(\frac{cp_l}{k_B T} (\psi_l + \chi + \zeta_l - \zeta_l \chi) \right) - 1 \right\}^{-1}, \tag{28}$$

which is obtained by substitution of (1) and (27) into (26). Here $\zeta_l = 1 - \frac{\mathbf{p}_l \mathbf{N}}{p_l}$.

Taking into account (23) and (26) N_b and N_d can be written as

$$N_d = n_1 \nu_{1 \to 3}, \tag{29}$$

$$N_b = \exp \left(-\frac{\varepsilon_1 - \mathbf{p}_1 \mathbf{u}}{k_B T} \right) (1 + n_1) \nu_{1 \to 3}, \tag{30}$$

where

$$\nu_{1 \to 3} = \frac{1}{3!} \int W(\mathbf{p}_1 | \mathbf{p}_2 \mathbf{p}_3 \mathbf{p}_4) \left(1 + n_2^{(0)} \right) \left(1 + n_3^{(0)} \right) \left(1 + n_4^{(0)} \right)$$
$$\times \delta \left(\varepsilon_{\sum} \right) \delta \left(\mathbf{p}_{\sum} \right) d^3 p_2 \, d^3 p_3 \, d^3 p_4. \tag{31}$$

So in order to find N_b and N_d, which are in the kinetic equation, it is necessary to find $\nu_{1 \to 3}$.

For small deviations from the equilibrium state of phonons with momentum $\mathbf{p}_1$, the distribution functions are equal to

$$n_1 = n_1^{(0)} + \delta n, \qquad n_2 = n_2^{(0)}, \qquad n_3 = n_3^{(0)}, \qquad n_4 = n_4^{(0)}. \tag{32}$$

Substituting (32) into (22) taking into account (31), we get

$$\frac{d\delta n}{dt} = -\frac{\delta n}{\tau_{1 \to 3}}, \tag{33}$$

where

$$\tau_{1 \to 3} = \frac{1 + n_1^{(0)}}{\nu_{1 \to 3}} \tag{34}$$

is a typical time of establishment of equilibrium in the phonon system, caused by processes $1 \to 3$.

In the region of momentum (11) considered by us the value of $n_1^{(0)}$ is small. Thus, as follows from (34), the rate $\nu_{1 \to 3}$ actually coincides with the relaxation rate of a system due to one phonon decaying into three and combining three phonons into one.

4. The rate of processes for one phonon decaying into three phonons

The general expression for the rate $\nu_{1 \to 3}$ can be obtained by rewriting the expression (31), taking into account equations (13)–(16), in spherical coordinates

$$\nu_{1 \to 3} = \frac{p_1}{3 \times 2^{12} \pi^5 \hbar^7 \rho^2} \int M^2 \delta \left(\varepsilon_{\sum} \right) \delta \left(\mathbf{p}_{\sum} \right) \left(1 + n_2^{(0)} \right) \left(1 + n_3^{(0)} \right) \left(1 + n_4^{(0)} \right)$$
$$\times p_2^3 p_3^3 p_4^3 \, dp_2 \, d\varphi_2 \, d\zeta_2 \, dp_3 \, d\varphi_3 \, d\zeta_3 \, dp_4 \, d\varphi_4 \, d\zeta_4, \tag{35}$$

where $\zeta_i = 1 - \frac{\mathbf{p}_i \mathbf{N}}{p_i}$.

Without any restriction on generality we can choose the angle φ_1 as the computing origin of angle φ. Four δ-functions in the integrand of equation (35) allow us to make integration over φ_3, φ_4, ζ_4, p_4. As a result we have

$$\nu_{1 \to 3} = N p_1 \int \left\{ M_+^2 + M_-^2 \right\} \left(1 + n_2^{(0)} \right) \left(1 + n_3^{(0)} \right) \left(1 + n_4^{(0)} \right)$$

$$\times \frac{p_2^3 p_3^3 p_4^2}{\sqrt{R}} \, \mathrm{d}p_2 \, \mathrm{d}p_3 \, \mathrm{d}\zeta_2 \, \mathrm{d}\zeta_3 \, \mathrm{d}\varphi_2, \tag{36}$$

where

$$N = \frac{1}{3 \times 2^{11} \pi^5 \hbar^7 \rho^2 c}, \tag{37}$$

$$M_\pm = M \left(\cos \varphi_3 = \cos \varphi_3^{(\pm)}, \cos \varphi_4 = \cos \varphi_4^{(\pm)} \right), \tag{38}$$

$$\cos \varphi_3^{(\pm)} = \frac{(p_{1\perp} - p_{2\perp} \cos \varphi_2) \left(A + p_{3\perp}^2 - p_{4\perp}^2 \right) \mp p_{2\perp} \sin \varphi_2 \sqrt{R}}{2 A p_{3\perp}}, \tag{39}$$

$$\cos \varphi_4^{(\pm)} = \frac{(p_{1\perp} - p_{2\perp} \cos \varphi_2) \left(A - p_{3\perp}^2 + p_{4\perp}^2 \right) \pm p_{2\perp} \sin \varphi_2 \sqrt{R}}{2 A p_{4\perp}} \tag{40}$$

$$p_4 = p_1 - p_2 - p_3 + \phi, \qquad \zeta_4 = \frac{p_1 \zeta_1 - p_2 \zeta_2 - p_3 \zeta_3 + \phi}{p_4}, \tag{41}$$

$$A = p_{1\perp}^2 + p_{2\perp}^2 - 2 p_{1\perp} p_{2\perp} \cos \varphi_2, \qquad p_{i\perp} = p_i \sqrt{2 \zeta_i - \zeta_i^2}, \tag{42}$$

$$R = 4 p_{3\perp}^2 p_{4\perp}^2 - \left(p_{1\perp}^2 + p_{2\perp}^2 - p_{3\perp}^2 - p_{4\perp}^2 - 2 p_{1\perp} p_{2\perp} \cos \varphi_2 \right)^2. \tag{43}$$

The case of $\zeta_1 = 0$ is important, as then the expression (36) becomes much simpler because, firstly, the dependence of the matrix element on φ_3 and φ_4 vanishes and, secondly, the dependence of the integrand expression on φ_2 disappears and so the integration over φ_2 can be easily made analytically. Taking this into account we have

$$\nu_{1 \to 3}^{\theta_1 = 0} = 2 \pi N p_1 \int \frac{\left(M^{\theta_1 = 0} \right)^2 \left(1 + n_2^{(0)} \right) \left(1 + n_3^{(0)} \right) \left(1 + n_4^{(0)} \right)}{\sqrt{(\zeta_{3+} - \zeta_3)(\zeta_3 - \zeta_{3-})}}$$

$$\times \left(1 + \frac{2 p_1 p_2 \zeta_2}{(p_1 - p_2)^2} \right)^{-\frac{1}{2}} \frac{p_2^3 p_3^2 p_4^2}{p_1 - p_2} \, \mathrm{d}p_2 \, \mathrm{d}p_3 \, \mathrm{d}\zeta_2 \, \mathrm{d}\zeta_3, \tag{44}$$

where

$$M^{\theta_1 = 0} \equiv M_+ \left(\zeta_1 = 0 \right) = M_- \left(\zeta_1 = 0 \right), \tag{45}$$

$$\zeta_{3\pm} = \frac{b \pm 8 p_1 p_2^2 p_3 \sqrt{\zeta_2 (2 - \zeta_2)(\zeta_2 - \zeta_{2\min})(\zeta_{2\max} - \zeta_2)}}{2 |a|}. \tag{46}$$

Here

$$b = -8 p_1 p_2^2 p_3 \zeta_2^2 + 4 p_2 p_3 \left(2 p_2 p_3 - 2 p_1 p_4 + 2 p_1 \phi + 2 p_4 \phi - \phi^2 \right) \zeta_2$$
$$+ 4 p_3 \phi (p_1 - p_2)(2 p_4 - \phi), \tag{47}$$

$$a = -4 p_3^2 (p_1 - p_2)^2 \left(1 + \frac{2 p_1 p_2 \zeta_2}{(p_1 - p_2)^2} \right), \tag{48}$$

$$\zeta_{2\min} = -\frac{(2 p_3 - \phi)(2 p_4 - \phi)}{2 p_1 p_2}, \tag{49}$$

$$\zeta_{2\max} = \frac{(2 p_1 - 2 p_2 + \phi)\phi}{2 p_1 p_2}. \tag{50}$$

The expression (44) gives the maximum value of (36), the rate of decay of a phonon which moves along the anisotropy axis of the system, i.e. at $\theta_1 = 0$. It corresponds to the maximum stimulation of the small-angle process $1 \rightarrow 3$ by the anisotropic phonon system. This due to the integrand expression (44) containing the functions $n_2^{(0)}$, $n_3^{(0)}$, $n_4^{(0)}$, which stimulate the decay, so giving the maximum decay rate.

At $\chi \ll 1$, with increasing angle θ_1, the rate $\nu_{1\rightarrow3}(\theta_1)$ decreases; it tends to the minimal value which corresponds to the rate of phonon decay in a phonon vacuum, when there is no stimulation. The minimum value can be obtained from expression (44) in which the distribution functions $n_2^{(0)}$, $n_3^{(0)}$ and $n_4^{(0)}$ are put to zero.

It is easy to estimate the difference between the maximum and minimum values of the rate $\nu_{1\rightarrow3}$: it follows from simple estimations that the typical numerical values of distribution functions contained in the integrand expression (44) are about 0.1 in the conditions of the experiments [1–3] at $\theta_1 = 0$. In this case the maximum value of $\nu_{1\rightarrow3}^{\theta_1=0}$ is greater than the minimum by 1.3 times (see figure 2).

The expression (44) also gives the value of the rate of decay $1 \rightarrow 3$ in isotropic phonon systems; this obtained by putting the parameter $\chi = 1$ in (28).

5. An approximate expression for the rate $\nu_{1\rightarrow3}$

It is impossible to further integrate (44) exactly analytically. We will derive an approximation of the rate $\nu_{1\rightarrow3}$ from (44), which will allow us to find the dependences of the rate on the relevant parameters. For this we shall retain only the first three terms of (17), which, as was discussed above, are resonant. Then we can replace $M^{\theta_1=0}$ with M' in (44), which gives

$$M' = 4(u+1)^2 (m_2 + m_3 + m_4) = 4(u+1)^2 m. \tag{51}$$

Here

$$m_2 = -\frac{(p_1 - p_2)^2}{p_1 p_2} \frac{1}{\zeta_2 - \kappa_2}, \qquad \kappa_2 = \frac{p_1 - p_2}{p_1 p_2}(f_1 - f_2 - f_{1-2}), \tag{52}$$

$$m_3 = -\frac{(p_1 - p_3)^2}{p_1 p_3} \frac{1}{\zeta_3 - \kappa_3}, \qquad \kappa_3 = \frac{p_1 - p_3}{p_1 p_3}(f_1 - f_3 - f_{1-3}), \tag{53}$$

$$m_4 = -\frac{(p_1 - p_4)^2}{p_1 p_4} \frac{1}{\zeta_4 - \kappa_4}, \qquad \kappa_4 = \frac{p_1 - p_4}{p_1 p_4}(f_1 - f_4 - f_{1-4}), \tag{54}$$

where κ_i are functions depending on a dispersion, and $f_{i-j} = f(p_i - p_j)$.

The integration in (44) can be made if we omit, in the integrand expression, distribution functions which are small, and replace weakly changing functions of momenta and angles by their typical values. As a result we have

$$\nu_{1\rightarrow3} = N'\bar{m}^2 \left(p_{2,3,4} = \frac{p_1}{3}\right) \phi \left(p_{2,3,4} = \frac{p_1}{3}\right)$$
$$\times \left(p_{\max}^2 - p_1^2\right)\left(12.5 p_1^6 - 13.5 p_1^4 p_c^2 + 2.43 p_c^6\right). \tag{55}$$

Here

$$N' = 6.69 \times 10^{-7} \frac{(u+1)^4}{\hbar^7 \rho^2 c}, \tag{56}$$

$$\bar{m} = m \left(\zeta_2 = 0, \zeta_3 = \frac{p_4 \phi}{2 p_3 (p_1 - p_2)}, \zeta_4 = \frac{(p_1 + p_3 - p_2) \phi}{2 p_4 (p_1 - p_2)}\right). \tag{57}$$
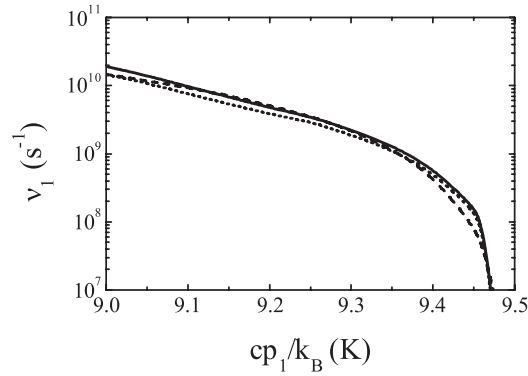
Figure 1. Momentum dependence of the rate $\nu_{1\to3}$ of one phonon decaying into three at $\theta_1 = 0$, calculated with a help of equation (44) (solid curve) at $\theta_1 = 180°$, with the help of equation (36) (dashed curve) and $\nu_{1\to3}$ with the help of equation (58) (dotted curve). All calculations were made with $\chi = 0.02$ and $T = 0.041$.
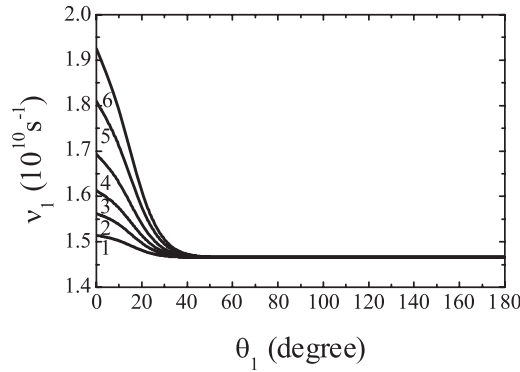

Figure 2. Angular dependence of the rate $\nu_{1\to3}$ of one phonon decaying into three for $\tilde{p}_1 = 9$ K, $\chi = 0.02$ and different values of temperature T such as 0.016 K, 0.021 K, 0.025 K, 0.030 K, 0.036 K, 0.041 K (curves 1–6 respectively).

The result (55) can be approximated with a simple expression which nevertheless represents the dependence of the rate on momentum p_1 and the parameters of superfluid helium

$$\nu_{1\to3} = (3.74 \times 10^{-5})\frac{(u+1)^4 p_c^5}{c\psi_{\max}\rho^2\hbar^7}\left(p_{\max}^2 - p_1^2\right)^2. \tag{58}$$

It follows from relation (58), that the factor $(p_{\max}^2 - p_1^2)^2$ determines the rapidity of the rate going to zero. Expression (58) does not retain the dependence on temperature because we omitted the distribution functions, which determine the temperature dependence of the rate $\nu_{1\to3}$, before the integration.

The dependences of $\nu_{1\to3}$ on p_1, ζ_1 and T can be obtained from numerical integration of expression (36).

In figure 1 the momentum dependence of the rate $\nu_{1\to3}$ of one phonon decaying into three is shown for $\theta_1 = 0$, calculated with equation (44) (solid curve), for $\theta_1 = 180°$ calculated with equation (36) (dotted curve) and $\nu_{1\to3}$ calculated with equation (58) (dashed curve). The calculations were made with the fixed values of $T = 0.041$ K and $\chi = 0.02$ which correspond to the conditions of experiments [1–3]. One can see that the rate rapidly decreases to zero
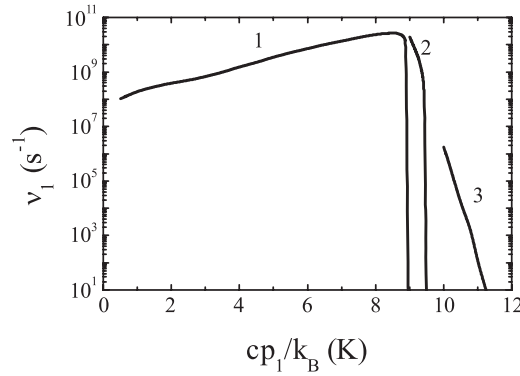
Figure 3. Momentum dependences of the rate of three-phonon processes $1 \rightarrow 2$ (curve 1), processes of one phonon decaying into three $1 \rightarrow 3$ (curve 2) and four-phonon processes $2 \rightarrow 2$ (curve 3). Calculations were made with $\theta_1 = 0$, $\chi = 0.02$ and $T = 0.041$.

at $cp_1/k_B = 9.49$ K, this is because at large values of momentum, processes of one phonon decaying into three are forbidden by the conservation laws. From equation (58) we see that the rate goes to zero as $(p_{max}^2 - p_1^2)^2$.

In figure 2 the angular dependence of the rate for one phonon decaying into three, at $\tilde{p}_1 = 9$ K, is shown for different temperatures. The rates are maximal at $\theta_1 = 0$. This is due to stimulated decay caused by the thermal phonons in the anisotropic system of phonons. With increasing angle, the rate becomes constant; this corresponds to spontaneous phonon decay, when phonons 'does not feel' the presence of the anisotropic system. When $\theta_1 = 0$ we see that on increasing the temperature from 0.016 K up to 0.041 K, the rate increases approximately by 25%, and at greater values of θ_1 the dependence on temperature disappears. The temperature dependence of the rate at $\theta_1 = 0$ is defined by the functions $n_i^{(0)}$ in the integrand of expression (44). At $\theta_1 \geqslant 40°$ the dependence of the rate on temperature disappears as the phonon 'does not feel' the presence of the anisotropic system when the angle is large.

The numerical calculations of the rate for the isotropic case with $T = 1$ K only differs from that for the anisotropic case, when $\theta_1 = 0$ and with typical values of $T = 0.041$ K and $\chi = 0.02$, by only a few per cent. That is why it is not shown here. The temperature dependence for the isotropic case is the same as for the anisotropic case with $\theta_1 = 0$.

As was stated in the introduction, the important question is where should we delimitate the l- and h-phonon subsystems. To answer this question we consider figure 3 in which are shown the momentum dependences of the rates for three-phonon processes (curve 1) (see [10]), the rates for one phonon decaying into three (curve 2) and the rates for four-phonon processes which create high-energy phonons with momenta $p > 10$ K (curve 3) (see [11]). Calculations were made with $\theta_1 = 0$, $\chi = 0.02$ and $T = 0.041$. From figure 3 it can be seen that the rates of three-phonon processes ($1 \rightarrow 2$) and processes of one phonon decaying into three ($1 \rightarrow 3$) are comparable and are much greater than the rates of four-phonon ($2 \rightarrow 2$) processes. This indicates that the range of momenta from 8.94 to 9.49 K should be included in the subsystem of low-energy phonons, in which equilibrium occurs rather quickly, and not in the region of high-energy phonons ($p > p_c$), where spontaneous decay processes are forbidden and the equilibrium is attained slowly.

There remains the question of which subsystem phonons with momenta from 9.49 to 10 K should they be assigned to. The numerical value of $\nu_{2 \rightarrow 2}(p_h)$, which describes the processes of creation and decay of high-energy phonons with momentum p_h due to processes $2 \rightarrow 2$ is
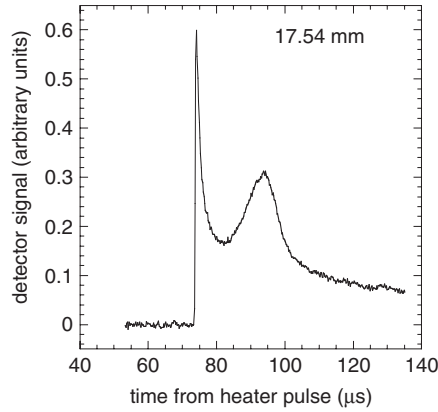
Figure 4. The l- and h-phonon signals due to a heater pulse at $t = 0$; the l-phonon signal starts at 73.6 μs and decays in a few μs, and the h-phonon signal starts at the time of the l-phonon signal and peaks at 94.1 μs. The signal at long times is due to heat leaking out of the glass heater–substrate. This curve is a detail from figure 1 in [3]. The phonon signal at 92.8 μs is due to 10.0 K phonons that have travelled from the heater to the detector at 189 m s^{-1}.

very sensitive to the numerical value of the momentum p_d, which separates l- and h-phonon subsystems. This is due to the strong anisotropy of phonon systems and the restrictions on angles between the interacting phonons (see [11]). As a result, the process of creation of an h-phonon from two interacting l-phonons is only effective when the momentum of one of the interacting phonons is close to p_d ($p_d <\sim p_c$). For example, an l-phonon pulse with typical experimental values of $\chi = 0.02$ and $T = 0.041$, [1–3], when $\tilde{p}_d = 10.0$ K, the rate $\nu_{2\rightarrow2}(10.01 \text{ K}) = 1.74 \times 10^6$ s^{-1}, and exponentially decreases with increasing p_h so that $\nu_{2\rightarrow2}(10.5 \text{ K}) = 1.77 \times 10^4$ s^{-1} and $\nu_{2\rightarrow2}(11 \text{ K}) = 1.01 \times 10^2$ s^{-1}. When $\tilde{p}_d = 9.49$ K the rate $\nu_{2\rightarrow2}(9.5 \text{ K}) = 3.6 \times 10^6$ s^{-1} and $\nu_{2\rightarrow2}(10 \text{ K}) = 5.91 \times 10^4$ s^{-1}. When $\tilde{p}_d = 8.94$ K the rate $\nu_{2\rightarrow2}(8.95 \text{ K}) = 1.04 \times 10^7$ s^{-1}, $\nu_{2\rightarrow2}(9.5 \text{ K}) = 2.13 \times 10^5$ s^{-1} and $\nu_{2\rightarrow2}(10 \text{ K}) = 3.23 \times 10^3$ s^{-1}. From these numerical values we see that the dependence of ν_1 on p_h shifts to smaller momenta, in a parallel way, with decreasing p_d.

The calculation of the rates of decay in the range of momenta from 9.49 to 10 K is very complicated as the order of integrals strongly increases with the number of phonons in the process. It is possible only to argue that the rates of decay are sufficiently rapid that the time to establish equilibrium in the region from 9.49 K up to 10 K will be less than in the region of $p > p_c$ where spontaneous decay processes are forbidden. We also note that in this momentum region, the rate of four-phonon processes $\nu_{2\rightarrow2}$ will be greater than in the region $\tilde{p}_1 > 10$ K. This leads to the conclusion that the total relaxation rate, which is the sum of the rates of four-phonon processes and decay processes, will surely be greater in the region from 9.49 to 10 K than in the region $\tilde{p}_1 > 10$ K.

This conclusion is supported by experiments. In figure 4 we show a phonon pulse after propagating 17.54 mm in liquid ^{4}He at 50 mK; this is a detail from figure 1 in [3]. The narrow l-phonons clearly arrive before the dispersed h-phonons. The propagation distance is found from the start time of the l-phonon signal and the velocity of sound (238.8 m s^{-1}). The h-phonon pulse shape is broad for two reasons; the first is that phonons are created in the liquid helium over many millimetres from the heater, and secondly because phonons with momentum higher than $cp/k_B = 10.0$ K are created albeit with diminishing probability as momentum increases. The h-phonons that arrive earlier than the peak are predominantly phonons with momentum around 10 K but created at different points in the liquid, and the h-phonons that arrive after the

peak are higher-momentum phonons and so have a lower group velocity. It should be noted that the signal which arrives later than $\approx 100\ \mu$s is not wholly due to h-phonons but is mainly caused by heat slowly leaking from the glass heater–substrate after the heater pulse. The momentum of a phonon created near the heater and travelling for $100\ \mu$s is $cp/k_B = 10.75$ K; we believe that there are few phonons with $cp/k_B > 11$ K.

The group velocity of the h-phonons at the peak is 186.4 m s^{-1} using the fact that the highest creation rate of h-phonons is near the heater [22, 23]. From Stirling's neutron scattering data [12] we derive the group velocity as a function of momentum and find that 186.4 m s^{-1} corresponds to $cp/k_B = 10.15$ K. This is sufficiently close to $cp/k_B = 10.0$ K, with group velocity 189 m s^{-1}, to confirm that the boundary is at $\tilde{p}_c = 10.0$ K.

6. Conclusion

In this paper the processes of one phonon decaying into three, both in anisotropic and isotropic phonon systems of superfluid helium, were investigated. The restrictions on the momentum of phonons which can participate in these processes were found, and the probability density of transition caused by the above-mentioned process was calculated (13)–(20).

The general expression for the rate $\nu_{1\to3}$ of one phonon decaying into three was found (36). Together with (29), (30) and (34), $\nu_{1\to3}$ describes the process of establishing equilibrium in anisotropic and isotropic phonon systems in the energy range where processes of one phonon decaying into two are forbidden.

The rate $\nu_{1\to3}$ in isotropic phonon systems, given by expression (44), was obtained from the general expression (36), and also the maximum and minimum values of the rate $\nu_{1\to3}$ in anisotropic phonon systems. Starting from the expression (44) a useful approximation was obtained which is given in (58). This shows the explicit dependence of the rate $\nu_{1\to3}$ on momentum (see figures 1 and 2). The dependence of the rate $\nu_{1\to3}$ on all parameters was found from expressions (36), (44) and (58); the physical reasons for these dependences were discussed.

The rate $\nu_{1\to3}$ is high (see figure 3), so equilibrium is rapidly established in the phonon system. This means that phonons with momenta up to 9.49 K should be considered as the low-energy phonon subsystem, in which equilibrium occurs quickly, in contrast to the region of high-energy phonons ($p \geqslant p_c$) where decay processes are forbidden and equilibrium is attained slowly.

We further argued that in the momentum range from 9.49 to 10.0 K that the total relaxation rate is necessarily higher than in the range above 10 K. Hence we conclude that the boundary between the l- and h-phonons is at momentum $p = p_c$. So at zero pressure, phonons with momenta $cp/k_B \leqslant 10$ K are l-phonons and those with momenta $cp/k_B > 10$ K are h-phonons. At higher pressures we must use the pressure dependent values of p_c, as was found experimentally [2]. This boundary has been used in calculations in [4, 5, 11]. This theoretical conclusion is supported by experiments [2, 3].

Acknowledgment

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